

Magnetic translation group as group extensions

WOJCIECH FLOREK

A. Mickiewicz University, Institute of Physics

Extensions of a direct product T of two cyclic groups \mathbb{Z}_{N_1} and \mathbb{Z}_{N_2} by an Abelian (gauge) group G with the trivial action of T on G are considered. All possible (nonequivalent) factor systems are determined using the Mac Lane method. Some of resulting groups describe magnetic translation groups. As examples extensions with $G = U(1)$ and $G = \mathbb{Z}_N$ are considered and discussed.

1 Introduction

The idea of magnetic translation groups, appearing in considerations of movement of electron in an external magnetic field, was proposed independently by Brown [1] and Zak [2]. From those works follows that the magnetic translation group is an image of Weyl–Heisenberg group [3] obtained by imposing the Born–von Kármán periodic boundary conditions. The general description of similar problems has been presented by Schwinger [4], who considered *unitary operator bases*. One of the considered cases can be interpreted as a description of *finite phase space*. This two-dimensional space is spanned by one space (positional) dimension and the other corresponding to kinetic momentum. Two unitary translation operators acting in these two dimensions are given by exponential functions of Hermitian operators of momentum and position, respectively, and, of course, they do not commute. Algebraic structure generated by such operators resembles an extension of a direct product of two Abelian (translation) groups by a group G containing commutators (or simply factors since, in general, it is a subgroup of the field of complex numbers). Therefore, such extensions are throughout studied in this work and the physical relevance is indicated.

In the next section some basic ideas of the Weyl–Heisenberg group, finite phase spaces and magnetic translations groups are briefly presented. All possible (central) extensions of a direct product T of two finite cyclic groups \mathbb{Z}_{N_1} and \mathbb{Z}_{N_2} by an Abelian group G are determined in Sec 3. The Mac Lane method [8, 9] (see also Lulek [10, 11, 12]), has been applied and the solution can be given in a general (analytic) form (see also the appendix). In Sec 4 the cases of $G = U(1)$ and $G = \mathbb{Z}_N$ are discussed as the simplest, and the most important, examples of possible groups of factors (gauge groups). This work is ended by a short discussion and remarks.

2 Basic Ideas

2.1 Weyl–Heisenberg Group

Let Q and P be two Hermitian canonically conjugated operators, *ie* a *complementary* pair of operators, *viz*

$$[Q, P] = i\hbar. \quad (1)$$

It is natural to transfer this property to the unitary operators, which are more accessible than the Hermitian ones. As a rule one constructs (unitary) operators using the exponential function

$$\mathcal{Q} = \exp(iQ\alpha) \quad \text{and} \quad \mathcal{P} = \exp(iP\beta), \quad (2)$$

where α and β are real numbers (parameters). A group generated by these operators is non-Abelian one since from the above formulae one immediately obtains [3, 5]

$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} \exp(i\alpha\beta\hbar). \quad (3)$$

The Born–von Kármán periodic conditions (the same period N for both operators) yield a finite group generated by two operators U and V with the following relations

$$UV = VU\varepsilon, \quad U^N = V^N = E, \quad \varepsilon^N = 1, \quad (4)$$

so ε is the N -th root of $1 \in \mathbb{C}^*$. This group, roughly speaking, is a *magnetic translation group* (*cf* [6]). The relation (3) can be considered as a basis for the Weyl algebra [3, 5] and its finite counter-part (4) has been investigated by Schwinger [4].

2.2 Finite Phase Space

Let us summarize the most important (in our considerations) results of Schwinger's work [4]. In a finite-dimensional eigenspace of a given Hermitian operator (say P) with a basis $\{|i\rangle \mid 0 \leq i < N\}$ a unitary operator of the cyclic permutation can be introduced, *ie*

$$V|i\rangle = |i + 1 \bmod N\rangle, \quad (5)$$

so, in general,

$$V^n|i\rangle = |i + n \bmod N\rangle \quad (6)$$

and

$$V^N|i\rangle = |i\rangle, \quad \text{so} \quad V^N = E \quad (7)$$

(E is the identity operator in the N -dimensional space). The eigenvalues of V obey the same equation, *ie* $v_k^N = 1$, and they are given by the N distinct complex numbers

$$v_k = \varepsilon^k, \quad k = 0, 1, \dots, N-1, \quad (8)$$

where $\varepsilon = \exp(2\pi i/N)$. The corresponding eigenvectors are given as the following linear combinations

$$|k\rangle_V = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \varepsilon^{-ki} |i\rangle, \quad (9)$$

so $V|k\rangle_V = v_k|k\rangle_V$, and this equation is simply a finite version of the Fourier transformation. In the new basis $\{|k\rangle_V \mid 0 \leq k < N\}$ one can define another unitary operator U by the ('anty'-)cyclic permutation, *ie*

$$U|k\rangle_V = |k - 1 \bmod N\rangle_V. \quad (10)$$

The properties of this operator are the same as for V , but now the eigenvectors are given as

$$|l\rangle_U = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \varepsilon^{lk} |k\rangle_V. \quad (11)$$

Combining (9) and (11) one obtains

$$|l\rangle_U = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{k(i-l)} |i\rangle = |l\rangle, \quad (12)$$

what can be easily presumed since (11) describes the inverse of the (finite) Fourier transformation (9). The considered system of (unitary) operators and their eigenvectors fulfil the following conditions

$$\begin{aligned} {}_V\langle k|l\rangle_U &= \frac{1}{\sqrt{N}} \varepsilon^{kl}, & {}_U\langle l|k\rangle_V &= \frac{1}{\sqrt{N}} \varepsilon^{-kl}, \\ V^N &= E, & U^N &= E, \\ UV &= VU\varepsilon, & U^l V^k &= V^k U^l \varepsilon^{kl}. \end{aligned} \quad (13)$$

Some properties of such a system and the physical relevance are discussed in more details by Schwinger [4]. For our aim it is important that it can be described as an extension of groups. It should be stressed that though $\varepsilon \in U(1) \subset \mathbb{C}^*$ we can limit ourselves to the cyclic (multiplicative) group $C_N \simeq \mathbb{Z}_N$ generated by the primitive N -th root of 1 $\in \mathbb{C}^*$.

2.3 Magnetic Translation Group

In the case of 2-dimensional magnetic translation groups the roles of the Hermitian operators Q and P are played by the operators π_{cx} and π_{cy} , which are connected with coordinates of the center of the magnetic orbit, *ie* the orbit of an electron in a magnetic field (see [6] and [7] for details). Strictly speaking, there are the following operators (\vec{H} is a uniform magnetic field along the z axis):

$$\vec{\pi} = \vec{p} + \frac{e}{2c}(\vec{H} \times \vec{r}), \quad \text{kinetic momentum,} \quad (14)$$

$$\vec{\pi}_c = \vec{p} - \frac{e}{2c}(\vec{H} \times \vec{r}), \quad \text{center of magnetic orbit,} \quad (15)$$

$$\left. \begin{aligned} T_x(a) &= \exp\left(\frac{i}{\hbar}\pi_{cx}a\right) \\ T_y(b) &= \exp\left(\frac{i}{\hbar}\pi_{cy}b\right) \end{aligned} \right\} \quad \text{magnetic translations} \quad (16)$$

and

$$Q = y_0 = \pi_{cx} \frac{c}{eH}, \quad (17)$$

$$P = -\frac{eH}{c}x_0 = \pi_{cy}, \quad (18)$$

where the pair (x_0, y_0) gives the coordinates of the center of the magnetic orbit (see [7] and references quoted therein). The parameters a and b determine the exponential transformation (16) and, after imposing the Born-von Kármán periodic conditions, correspond to lattice constants. The last pair of Hermitian operators preserves the commutation relations between the position and momentum coordinates, *ie*

$$QP - PQ = i\hbar \quad (19)$$

and their images under transformation (2) are the magnetic translations (16), *viz*

$$\begin{aligned} \mathcal{Q} = \exp(iQ\alpha) &= T_x\left(\frac{\alpha\hbar c}{eH}\right), \\ \mathcal{P} = \exp(ip\beta) &= T_y(\beta\hbar). \end{aligned} \quad (20)$$

3 Group Extensions

The above presented brief summary of the most important results for the magnetic translation group and a pair of complementary operators indicates that magnetic translation groups can be described as an extension of a direct product of two cyclic groups (of order N and generated by \mathcal{Q} and \mathcal{P} , respectively) by a ‘factor’ (or ‘gauge’) group G being a subgroup of the multiplicative group \mathbb{C}^* . It should be stressed that the term *translation group* is a little misleading, since the unitary operators T_x and T_y do not commute and they rather correspond to a two-dimensional phase space with one direction connected with a position and the second one with a momentum (so it is a pair of a one-dimensional space L and its adjoint L^* rather than a product $L \otimes L$). However, the algebraic structure of the translation group does not depend on it because $L \cong L^*$. Therefore in this section we investigate a general problem of finding all non-equivalent extensions $G \square (\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2})$. We assume G to be an Abelian group due to the physical relevance ($\varepsilon \in \mathbb{C}^*$) on the one hand, and, on the other hand, due to some mathematical problems connected with non-Abelian extensions. The physical applications suggest that the trivial action of T on G should be considered, so — strictly speaking — a central extension $G \bigcirc (\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2})$ is investigated. The problem has been solved applying the Mac Lane method [8, 9] described also by Lulek [10] (for more details see the review articles [11, 12] and references quoted therein).

3.1 Alphabets and the Schreier Set

A two-dimensional finite translation group T will be hereafter consider as a direct product of two cyclic groups \mathbb{Z}_{N_i} , $N_i > 1$, $i = 1, 2$. Therefore,

$$T = \{t = (t_1, t_2) \mid t_i \in \mathbb{Z}_{N_i}, i = 1, 2\}. \quad (21)$$

As a set of generators one can choose pairs

$$A \equiv \{\tau_1 := (1, 0), \tau_2 := (0, 1)\}. \quad (22)$$

In the Mac Lane method the second cohomology group $H^2(T, G)$, describing all non-equivalent extensions of T by G , can be found after considerations of a free group F and its normal subgroup R — a kernel of the homomorphism $M : F \rightarrow T$. Moreover, one has to study the so-called operator homomorphisms $\phi : R \rightarrow G$ and crossed homomorphisms $\gamma : F \rightarrow G$. In order to do it the alphabets X (of F) and Y (of R) have to be found.

Let F be a free group such that there exists a homomorphism $M : F \rightarrow T$ and $M(X) = A$, where X is an alphabet of F . Of course, X consists of two letters, say x_1 and x_2 , with $M(x_i) = \tau_i$, $i = 1, 2$. For any n -letter word $F \ni f = \prod_{i=1}^n \alpha_i^{\varepsilon_i}$, where $\alpha_i \in X$ and $\varepsilon_i = \pm 1$, one obtains

$$M(f) = \left(\sum_{i \in E_1} \varepsilon_i \right) \bmod N_1, \left(\sum_{i \in E_2} \varepsilon_i \right) \bmod N_2. \quad (23)$$

Subsets E_j consist of indices $1 \leq i \leq n$ such that $\alpha_i = x_j$ for $j = 1, 2$. Eg, for $N_1 = 5, N_2 = 6$

$$M(x_1^2 x_2^{-1} x_1^{-1} x_2^4 x_1^7 x_2^{-5} x_1) = (9 \bmod 5, -2 \bmod 6) = (4, 4).$$

The kernel $\text{Ker } M \triangleleft F$, denoted hereafter as R , corresponds to group relations imposed on generators x_1 and x_2 . As representatives of *right* cosets $R \setminus F$ the following elements are chosen

$$f_{(t_1, t_2)} = \Psi(t_1, t_2) := x_1^{t_1} x_2^{t_2}, \quad (24)$$

where $\Psi : T \rightarrow F$ is such a mapping that $M \circ \Psi = \text{id}_T$. This mapping determines also a choice function

$$\beta := \Psi \circ M, \quad \beta : F \rightarrow F, \quad (25)$$

which maps each element $f \in F$ onto the corresponding (right-)coset representative f_t , where $t = M(f)$. These elements form the so-called Schreier set

$$S := \{x_1^{t_1} x_2^{t_2} \mid t_i \in \mathbb{Z}_{N_i}, i = 1, 2\}. \quad (26)$$

In general, the coset representatives do not form a group, but

$$f_t f_{t'} = \rho(t, t') f_{tt'}, \quad (27)$$

where $\rho(t, t') \in R$. These elements determine a factor system $m : T \otimes T \rightarrow G$, then a group extension $G \bigcirc T$ via an operator homomorphism (see below).

The alphabet Y of the kernel R can be chosen as nontrivial different factors $\rho(t, t') = f_t f_{t'} f_{tt'}^{-1}$ for $t \in T$ and $t' \in A$. According to the Nielsen–Schreier theorem there are

$$|Y| = 1 + (|X| - 1)|T| \quad (28)$$

letters in this alphabet, so (in the considered case) one obtains $|Y| = N_1 N_2 + 1$. It is straightforward matter to show that these letters are given by the following formulae

$$\begin{aligned} A_{t_2} &= x_1^{N_1-1} x_2^{t_2} x_1 x_2^{-t_2} && \text{for } 0 \leq t_2 < N_2, \\ B_{t_1} &= x_1^{t_1} x_2^{N_2} x_1^{-t_1} && \text{for } 0 \leq t_1 < N_1, \\ C_{t_1 t_2} &= x_1^{t_1} x_2^{t_2} x_1 x_2^{-t_2} x_1^{-t_1-1} && \text{for } 0 \leq t_1 < N_1 - 1 \text{ and } 1 \leq t_2 < N_2. \end{aligned} \quad (29)$$

All factors $\rho(t, t')$ can be written in this alphabet since they are elements of the kernel R . Let $f \in R$ be a word given in the alphabet X . To find out its ‘spelling’ in the alphabet Y one can use the ‘translation’ formula

$$f = \prod_{i=1}^n \alpha_i^{\varepsilon_i} = \prod_{i=1}^n \beta(f_{i-1}) \alpha_i^{\varepsilon_i} \beta(f_{i-1} \alpha_i^{\varepsilon_i})^{-1}, \quad (30)$$

where f_i denotes an initial subword of f consisting of the first i letters ($f_0 := 1_F$). Each nontrivial factor in the above product is either a letter of the alphabet Y or the inverse of a letter (ie an element of Y^{-1}). Introducing a new set of letters

$$D_{t_1 t_2} := \prod_{i=0}^{t_1-1} C_{it_2}, \quad D_{0 t_2} := 1_F, \quad (31)$$

all factors $\rho(t, t')$ can be written as

$$\rho((t_1, t_2), (t'_1, t'_2)) = D_{t_1 t_2}^{-1} (D_{N_1 t_2} A_0)^{e_1} D_{t'_1 t_2} (D_{t'_1 N_2} B_0)^{e_2} \quad (32)$$

where

$$t''_i = t_i + t'_i \bmod N_i = t_i + t'_i - e_i N_i \quad (33)$$

and

$$e_i = \begin{cases} 1 & \text{if } t_i + t'_i \geq N_i, \\ 0 & \text{otherwise.} \end{cases}$$

Other properties of the letters A, B, C , and D are gathered in the appendix.

Table 1: Action of X on Y

y	$x_1yx_1^{-1}$	$x_2yx_2^{-1}$
A_{t_2}	$A_0C_{0t_2}$	$D_{N_1-1,1}A_{t_2+1}$
B_{t_1}	B_{t_1+1}	$D_{t_11}B_{t_1}D_{t_11}^{-1}$
$C_{t_1t_2}$	C_{t_1+1,t_2}	$D_{t_11}C_{t_1t_2+1}D_{t_1+1,1}^{-1}$

3.2 Operator and Crossed Homomorphisms

An operator homomorphism $\phi : R \rightarrow G$ fulfils the condition

$$\phi(frf^{-1}) = (\Delta \circ M)(f)(\phi(r)), \quad (34)$$

where $\Delta : T \rightarrow \text{Aut } G$ describes an action of T on G . For the trivial action ($\Delta(f) = \text{id}_G$) one obtains

$$\phi(frf^{-1}) = \phi(r). \quad (35)$$

Each homomorphism ϕ is determined by its values for $f = x \in X$ and it is enough to consider $r = y \in Y$. The elements xyx^{-1} , $x \in X$, $y \in Y$ are gathered in Table 1. The set of equations (35), solved in an Abelian group G , provides us with the following conditions

$$\begin{aligned} a_{t_2} &= a_0 + c_{0t_2}, & a_{t_2} &= a_{t_2+1} + d_{N_1-1,1}, \\ b_{t_1} &= b_{t_1+1}, & b_{t_1} &= b_{t_1}, \\ c_{t_1t_2} &= c_{t_1+1,t_2}, & c_{t_1t_2} &= c_{t_1,t_2+1} - c_{t_11}, \end{aligned} \quad (36)$$

where the lower-case letters denote images of the upper-case letters: $a_{t_2} = \phi(A_{t_2})$, $b_{t_1} = \phi(B_{t_1})$, $c_{t_1t_2} = \phi(C_{t_1t_2})$. The solution can be written as

$$a_{t_2} = a + t_2c, \quad b_{t_1} = b, \quad c_{t_1t_2} = t_2c, \quad (37)$$

so $d_{t_1t_2} = \phi(D_{t_1t_2}) = t_1t_2c$. The parameters $a \equiv a_0$ and $b \equiv b_0$ are any elements of the factor group G , but the parameter $c \in G$ fulfils the condition

$$N_1c = N_2c = 0. \quad (38)$$

Therefore, non-trivial solutions for c exist if and only if $\gcd(N_1, N_2) = M \neq 1$ and there is an element $g \in G$ with order dividing M . It means that nontrivial solutions of (38) are possible if and only if G has a torsion subgroup.

A crossed homomorphism γ is determined as a mapping satisfying the following condition

$$\gamma(f, f') = \gamma(f) + (\Delta \circ M)(f)(\gamma(f')).$$

Therefore, in the considered case, the crossed homomorphisms $\gamma : F \rightarrow G$ become ‘ordinary’ ones and they are determined by their values ξ_1, ξ_2 for the letters $x_1, x_2 \in X$, respectively. For letters in the alphabet Y one immediately obtains that

$$\gamma(A_{t_2}) = N_1\xi_1, \quad \gamma(B_{t_1}) = N_2\xi_2, \quad \gamma(C_{t_1t_2}) = \gamma(D_{t_1t_2}) = 0. \quad (39)$$

It is evident that for a torsion-free group G the non-zero values ξ_i , $i = 1, 2$, yield the non-zero values of $\gamma(A_{t_2})$ and $\gamma(B_{t_1})$, but when there are the elements with orders being divisors of N_1 or/and N_2 then they can give $\gamma(A_{t_2})$ or/and $\gamma(B_{t_1})$ equal to 0.

The second cohomology group $H^2(T, G)$ can be found as a quotient group of the group of operator homomorphisms $\text{Hom}_F(R, G)$ and the group of crossed homomorphisms $Z^1(F, G)$ restricted to R . Therefore, one has to find those $\phi \in \text{Hom}_F(R, G)$, which are not crossed homomorphisms. The first results is that nontrivial solutions of (38) lead to nontrivial operator homomorphisms, since for all crossed homomorphisms one has $\gamma(C_{t_1t_2}) = 0$. However, you are remembered that it is possible for a torsion or mixed group G only. Moreover, the parameters

a and b can take any value, whereas $\gamma(A_{t_2})$ and $\gamma(B_{t_1})$ are limited by the conditions (39). These questions can be answered more precisely when the arithmetic structure of numbers N_1 and N_2 are determined and the gauge group G is fixed. In the next section two, the most important, examples are presented.

A factor system $m : T \times T \rightarrow G$ can be found as images of factors $\rho(t, t')$ under nonequivalent operator homomorphisms ϕ . Therefore, a general formula for a factor system $m : T \times T \rightarrow G$ can be written as

$$m((t_1, t_2), (t'_1, t'_2)) = \phi(\rho((t_1, t_2), (t'_1, t'_2))) = e_1 a + e_2 b + t'_1 t_2 c. \quad (40)$$

This formula is discussed in the next section for $G = U(1)$ and $G = \mathbb{Z}_N$.

4 Examples

For the previous considerations follows that $\gcd(N_1, N_2) = M > 1$, so we can assume

$$N_1 = k_1 M, \quad N_2 = k_2 M, \quad \gcd(k_1, k_2) = 1. \quad (41)$$

4.1 Unitary Group $U(1)$

Let

$$G = U(1) = \{e^{i\delta} \mid 0 \leq \delta < 2\pi\} \quad (42)$$

(the multiplicative notion will be used hereafter what corresponds to the addition of arguments δ modulo 2π).

When one assumes $c = 1$ then all operator homomorphisms ϕ , determined by the parameters a and b , are crossed ones, since equations $a = \xi_1^{N_1}$ and $b = \xi_2^{N_2}$ can always be solved in $U(1)$. It immediately follows that from these considerations that a number of nonequivalent extensions is equal to a number of solutions of the condition (38). Then one has to find such l_1 and l_2 that

$$c = \exp(2l_1\pi i/N_1) = \exp(2l_2\pi i/N_2).$$

Therefore, the following condition must be satisfied

$$\frac{l_1}{k_1} = \frac{l_2}{k_2}; \quad 0 \leq l_i < Mk_i - 1$$

and one obtains

$$l_i = 0, k_i, 2k_i, \dots, (M-1)k_i.$$

As the final result all possible values of $\phi(C_{t_1 1}) = c$ are found as

$$c = \exp(2k\pi i/M) \quad \text{for } 0 \leq k < M. \quad (43)$$

It has been shown that there are $M = \gcd(N_1, N_2)$ nonequivalent extensions $U(1) \bigcirc (\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2})$. As a representative of equivalent factor systems this one with $a = b = 1$ can be chosen. It follows from the formula (40) that in this case the k -th factor system is given by the following equation

$$m_k((t_1, t_2), (t'_1, t'_2)) = \exp(2\pi ik t'_1 t_2 / M) = \omega^{t'_1 t_2 k}, \quad (44)$$

where $\omega = \exp(2\pi i/M)$ is the M -th root of 1. For any closed loop of translations $(0, N_2 - y), (N_1 - x, 0), (0, y)(x, 0)$ ($x, y \neq 0$) one immediately obtains

$$\begin{aligned} & [1, (0, N_2 - y)][1, (N_1 - x, 0)][1, (0, y)][1, (x, 0)] \\ &= [\omega^{xyk}, (N_2 - y, N_1 - x)][\omega^{xyk}, (x, y)] = [\omega^{xyk}, (0, 0)]. \end{aligned}$$

Therefore, a phase factor ω^{xyk} corresponds to such a loop. As an example in Table 2 the factor systems m_1 and m_2 for $N_1 = N_2 = 4$ are presented.

4.2 Finite Cyclic Groups

Let $G = \mathbb{Z}_N$ and $M = \gcd(N_1, N_2)$ (as above). Introducing $M' = \gcd(N, M)$ all integers N, N_1, N_2 can be written as

$$N = kk'_1 k'_2 M', \quad N_i = k_i k'_i k' M', \quad i = 1, 2,$$

Table 2: Factor systems for $U(1) \bigcirc (\mathbb{Z}_4 \otimes \mathbb{Z}_4)$; columns (rows) are labelled by t'_1 (t_2) only, since values of factors do not depend on t'_2 (t_1 , respectively)

		a) $k = 1$				b) $k = 2$					
t_2	t'_1	0	1	2	3	t_2	t'_1	0	1	2	3
0	0	1	1	1	1	0	0	1	1	1	1
1	1	1	\hat{i}	-1	$-\hat{i}$	1	1	-1	1	-1	
2	1	-1	1	1	-1	2	1	1	1	1	
3	1	$-\hat{i}$	-1	\hat{i}		3	1	-1	1	-1	

Table 3: Factor system for the extension $\mathbb{Z}_4 \bigcirc (\mathbb{Z}_4 \otimes \mathbb{Z}_4)$ with $a = 2$, $b = 3$, and $c = 1$; $t_1 t_2 = 00$ and $t'_1 t'_2 = 00$ are omitted

$t_1 t_2$	$t'_1 t'_2$	10	20	30	01	11	21	31	02	12	22	32	03	13	23	33
10	0	0	2	0	0	0	2	0	0	0	2	0	0	0	0	2
20	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	2
30	2	2	2	0	2	2	2	0	2	2	2	0	2	2	2	2
01	1	2	3	0	1	2	3	0	1	2	3	3	0	1	2	
11	1	2	1	0	1	2	1	0	1	2	1	3	0	1	0	
21	1	0	1	0	1	0	1	0	1	0	1	3	0	3	0	
31	3	0	1	0	3	0	1	0	3	0	1	3	2	3	0	
02	2	0	2	0	2	0	2	3	1	3	1	3	1	3	1	1
12	2	0	0	0	2	0	0	3	1	3	3	3	1	3	3	
22	2	2	0	0	2	2	0	3	1	1	3	3	1	1	3	
32	0	2	0	0	0	2	0	3	3	1	3	3	3	1	3	
03	3	2	1	3	2	1	0	3	2	1	0	3	2	1	0	
13	3	2	3	3	2	1	2	3	2	1	2	3	2	1	2	
23	3	0	3	3	2	3	2	3	2	3	2	3	2	3	2	
33	1	0	3	3	0	3	2	3	0	3	2	3	0	3	2	

where $M' = \gcd(N, N_1, N_2)$, $k' = \gcd(N_1, N_2)/M'$ and $k'_i = \gcd(N_i, N)/M'$. It follows from the condition (38) that

$$c = l k k'_1 k'_2 \quad \text{with} \quad 0 \leq l < M'. \quad (45)$$

Considering values of (crossed) homomorphisms one obtains that different values of $\gamma(A_{t_2}) = N_1 \xi_1$ and $\gamma(B_{t_1}) = N_2 \xi_2$ are obtained for kk'_2 and kk'_1 values of ξ_1 and ξ_2 , respectively. Therefore, nonequivalent extensions are determined by

$$0 \leq a < M' k'_1, \quad 0 \leq b < M' k'_2. \quad (46)$$

Hence, in the considered case there exist $(M')^3 k'_1 k'_2$ nonequivalent extensions with factor systems given by (40). Some of them are isomorphic, since when $a' = \alpha a$, $b' = \alpha b$ and $c' = \alpha c$, with $\alpha \in \mathbb{Z}_N$, then a mapping

$$[t, (t_1, t_2)] \mapsto [\alpha t, (t_1, t_2)] \quad (47)$$

determines a group isomorphism (if multiplication by α is an automorphism of \mathbb{Z}_N , ie if and only if $\gcd(\alpha, N) = 1$). Of course, other isomorphisms can also be found.

It is evident that for all $c \neq 0$ each closed loop with mixed translations in the first and second directions gains an appropriate factor $xyc \bmod N$. But in this case also loops along x -th or y -th direction gain a factor connected with the other parameters a and b , viz

$$N_1[0, (1, 0)] = [0, (1, 0)] + \dots + [0, (1, 0)] = [0, (N_1 - 1, 0)] + [0, (1, 0)] = [a, (0, 0)].$$

From general properties of finite groups (especially abelian and cyclic ones) follows that it is enough to consider a case when all numbers N , N_1 and N_2 are powers of a prime integer p . In general, there are possible three cases

$$\begin{aligned} \text{(I)} : \quad N &= p^\alpha, & N_1 &= p^{\alpha+\beta+\gamma}, & N_2 &= p^{\alpha+\gamma}; \\ \text{(II)} : \quad N &= p^{\alpha+\beta}, & N_1 &= p^{\alpha+\beta+\gamma}, & N_2 &= p^\alpha; \\ \text{(III)} : \quad N &= p^{\alpha+\beta+\gamma}, & N_1 &= p^{\alpha+\beta}, & N_2 &= p^\alpha. \end{aligned} \quad (48)$$

Table 4: The group of operator homomorphisms $\phi : R \rightarrow \mathbb{Z}_2$

	y_1	y_2	y_3	y_4	y_5	abc
ϕ_1	0	0	0	0	0	000
ϕ_2	0	0	1	1	0	001
ϕ_3	0	1	0	0	1	010
ϕ_4	0	1	1	1	1	011
ϕ_5	1	0	0	1	0	100
ϕ_6	1	0	1	0	0	101
ϕ_7	1	1	0	1	1	110
ϕ_8	1	1	1	0	1	111

In all cases b and c has a value from the set $\{0, 1, \dots, p^\alpha - 1\}$ but $a = 0, 1, \dots, p^\alpha - 1$ only in the case (I). In the two other cases $a = 0, 1, \dots, p^{\alpha+\beta} - 1$. So, a number of nonequivalent extension is $p^{3\alpha}$ in the first case and $p^{3\alpha+\beta}$ in the second and third cases. It should be underlined that these results do not depend on γ . The special case $\alpha = 0$ yields a direct product in the first case and extensions of two cyclic groups in the other cases. On the other hand, for all cases (I), (II) and (III) the condition $\beta = \gamma = 0$ gives the same type of extensions, *viz* $\mathbb{Z}_{p^\alpha} \bigcirc (\mathbb{Z}_{p^\alpha} \otimes \mathbb{Z}_{p^\alpha})$ (number of nonequivalent extension is, of course, equal to $p^{3\alpha}$). As an example the case $p^\alpha = 4$ for $a = 2$, $b = 3$, and $c = 1$ is presented in Table 3.

4.3 Classification of extensions

The parameters a , b , and c provide a classification scheme of all nonequivalent extensions. The most rough way is to distinguish zero and non-zero values of these parameters, what yields eight types of extensions. Due to possible isomorphisms a number of completely different (*ie* non-isomorphic) extensions is less than 8. It can be shown considering the simplest example $\mathbb{Z}_2 \bigcirc (\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathbb{Z}_2 \bigcirc D_2$, which has been studied in [12]. In the presented considerations the parameter c plays a special role, since:

- It leads to noncommutativity of group elements in the extension, whereas for $c = 0$ the obtained groups are Abelian ones;
- The two other parameters correspond to full loops along x and, respectively, y axes, whilst c is connected with a one-square loop;
- When a gauge group G is assumed to be continuous one then the parameters a and b lead to trivial factors.

Therefore, in the first step all (nonequivalent) extensions can be divided into Abelian ($c = 0$) and non-Abelian ($c = 1$) ones. Magnetic translation groups are non-Abelian, hence they can be found amongst groups of the second type.

In the above mentioned simplest case $\mathbb{Z}_2 \bigcirc D_2$ all nonequivalent extensions correspond to all eight types (1 is the unique non-zero element in \mathbb{Z}_2). As it has been shown in [12] there are five letters in the alphabet Y and they are given by the following letters A , B , and C introduced in this work

$$\begin{aligned} y_1 &= x_1^2 = A_0, & y_2 &= x_2^2 = B_0, & y_3 &= x_2 x_1 x_2^{-1} x_1^{-1} = C_{01} \\ y_4 &= x_1 x_2 x_1 x_2^{-1} = A_1, & y_5 &= x_1 x_2^2 x_1^{-1} = B_1. \end{aligned}$$

According with the formulae (37) one obtains

$$\phi(y_1) = a, \quad \phi(y_2) = b, \quad \phi(y_3) = c, \quad \phi(y_4) = a + c, \quad \phi(y_5) = b.$$

All possible operator homomorphism, denoted as ϕ_i with $i = 1, \dots, 8$, are presented in Table 4 rewritten from [12] (Table 5) and a column containing the triple abc is added.

Having this table and using the other results of [12] one can easily see that homomorphisms with odd indices yield Abelian extensions and that the first one is simply a direct product $\mathbb{Z}_2 \otimes D_2 \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \simeq D_{2h}$. The other three are isomorphic with a direct product $\mathbb{Z}_2 \otimes \mathbb{Z}_4$, but they differ in orders of elements $[0, (1, 0)]$, $[0, (0, 1)]$, and $[0, (1, 1)]$. For ϕ_3 (ϕ_5 and ϕ_7) order of the first (second and third, respectively) element is two, whilst it is 4 for the other two elements. For larger lattices such a classification is a bit more complicated due to many possible choices of generators for a direct product $\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2}$.

For non-Abelian extensions ($c = 1$, *i.e.* ϕ_i with even i) there are also two types: (i) ϕ_8 with $a = b = c = 1$ and the extension isomorphic with quaternion group (or double dihedral group D'_2 , *cf* [14, 15]) and (ii) extensions isomorphic with the dihedral group D_4 . Within the second type one deals with the same isomorphism caused by an arbitrary choice of generators for $\mathbb{Z}_2 \otimes \mathbb{Z}_2$: $\{(1, 0), (0, 1)\}$, $\{(1, 0), (1, 1)\}$, and $\{(0, 1), (1, 1)\}$ for ϕ_2 , ϕ_4 , and ϕ_6 , respectively. There are at least two facts indicating that this type of extensions corresponds to a magnetic translation group:

- The results should be the same (or, in a sense, similar) for a continuous group, *i.e.* one has to look for the magnetic translation group in types containing extensions with $a = b = 0$ and $c \neq 0$.
- It was shown in Sec 2.2 that unitary operators U and V have to fulfil condition $U^N = V^N = E$ (see Eq (13)), so — in the considered case — there should be in $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ two elements $(t_1, t_2) \neq (0, 0)$ such that $[0, (t_1, t_2)]^2 = [0, (0, 0)]$. It is evident that there are no such elements in D'_2 .

From this follows that the extension with $a = b = 0$ and $c = 1$. However, one has to remember that there are many possible choices of generators for $\mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2}$ and of \mathbb{Z}_N what leads to a class of isomorphic extensions.

5 Final Remarks

It has been shown that the Mac Lane method enables determination of all nonequivalent extensions of $T = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ by an Abelian (gauge) group G . Factor systems $m : T \times T \rightarrow G$ can be presented in an analytic form and they are parametrised by three elements $a, b, c \in G$. It is easy to notice that some of obtained extensions are isomorphic and this isomorphism is connected with an arbitrary choice of a generator of the cyclic group $C_N \subset U(1)$. This isomorphism directly corresponds to labelling of basis vectors in a finite dimensional space L (*cf* Sec 2.2) — ε may be not only the primitive root of 1 but also any power ε^k of the primitive root with $\gcd(k, N) = 1$. Moreover, the physical relevance indicates that extensions by $U(1)$ with k not mutually prime with M (*cf* Table 2b) should not be taken into account. This problem is discussed in more details by Walcerz [6] (see also references quoted therein). The main point is that the factor ε in Eq (13) has to be a generator of C_N . On the other hand, each linear representation of a magnetic translation group restricted to the factor subgroup G should be a faithful one.

From the considerations presented in this work there follows that the parameters a , b , and c correspond to full loops along the x and y axes and to a one ‘plaquette’ loop, respectively. The first two loops lead to the zero factors if a continuous group (*e.g.* $U(1)$) is assumed to be a gauge group, but the third one is always present and is relevant to a magnetic flux. The parameters a , b , and c provide a classification scheme of all nonequivalent extensions. The most important role is played by the parameter c and the extension with $a = b = 0$ and $c = 1$ can be chosen as a representative of a class of isomorphic extensions corresponding to magnetic translation groups.

This work should be completed by investigations of irreducible representations of studied extensions, especially of these ones, which describe magnetic translation groups. However, on the one hand, it is too cumbersome for a brief presentation and, on the other hand, it can be done applying the standard methods of induced and projective representations, which can be found in many monographs (*e.g.* [16]). Nevertheless, it can be noticed at the very first glance that representations with the physical relevance are obtained when a faithful representation of G is used in the induction procedure. In the considered case of finite gauge groups \mathbb{Z}_N it means that one should use one of the irreducible representation $\Gamma_l(k) = \exp(2\pi i k l / N)$ of \mathbb{Z}_N with $\gcd(l, N) = 1$. Different choices correspond, again, to different generators of \mathbb{Z}_N . It shows that for $N_1 = N_2 = N = 2$ the two-dimensional representation E of the dihedral group D_4 should be considered (*cf* [6]).

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Appendix: Relations in the Alphabet Y

The alphabet X of the free group F consists of two letters x_1 and x_2 such that $M(x_i) = \tau_i$, where τ_i , $i = 1, 2$, are generators of the translation group T (see Sec 3.1). Therefore,

$$M(x_i^{N_i}) = (0, 0) \quad \text{for } i = 1, 2, \tag{A.1}$$

so these two words, *viz*

$$A := x_1^{N_1}, \quad B := x_2^{N_2}, \tag{A.2}$$

belong to the kernel $\text{Ker } M \equiv R \triangleleft F$. According to the definition (23) each of words

$$C_{t_1 t_2} := x_1^{t_1} x_2^{t_2} x_1 x_2^{-t_2} x_1^{-t_1-1} \quad (\text{A.3})$$

belongs to R , too. For $t_1 > 0$ one can introduce words

$$D_{t_1 t_2} := \prod_{i=0}^{t_1-1} C_{it_2}. \quad (\text{A.4})$$

It is straightforward matter to show that

$$D_{t_1 t_2} := x_2^{t_2} x_1^{t_1} x_2^{-t_2} x_1^{-t_1} \quad (\text{A.5})$$

and, therefore,

$$x_2^{t_2} x_1^{t_1} = D_{t_1 t_2} x_1^{t_1} x_2^{t_2}. \quad (\text{A.6})$$

All letters of the alphabet Y (*cf* Eqs (29)) can be expressed using two letters (A.2) and $N_1 N_2 - 1$ (for $t_1 = 0, 1, \dots, N_1 - 1$ and $t_2 = 1, 2, \dots, N_2$ except for the pair $t_1 = N_1 - 1, t_2 = N_2$) letters C given by Eq (A.3). One can easily check that

$$A_{t_2} = C_{N_1-1, t_2} A, \quad (\text{A.7})$$

$$B_{t_1} = D_{t_1 N_2}^{-1} B. \quad (\text{A.8})$$

From the above formulae and Eqs (29) follows that

$$\begin{aligned} A_0 &= A, & A_{N_2} &= B_{N_1-1} A B^{-1}, \\ B_0 &= B, & B_{N_1} &= A B A^{-1}, \\ C_{t_1 0} &= 1_F, & C_{t_1 N_2} &= B_{t_1} B_{t_1+1}^{-1}. \end{aligned}$$

For example, these relations yield

$$B_2 = (C_{0 N_2} C_{1 N_2})^{-1} B = (x_1^2 x_2^{N_2} x_1^{-2} x_2^{-N_2}) x_2^{N_2} = x_1^2 x_2^{N_2} x_1^{-2}.$$

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